Logarithmic Voronoi polytopes for discrete linear models

Yulia Alexandr (UC Berkeley)

Algebraic Statistics 2022
University of Hawai'i at Manoa
May 19, 2022

Discrete linear models

A *linear model* is given parametrically by nonzero linear polynomials.

Theorem (A., Heaton, 2021)

Let $\mathcal{M}$ be a linear model. Then the logarithmic Voronoi cells are equal to their log-normal polytopes.

Discrete linear models

A *linear model* is given parametrically by nonzero linear polynomials.

Theorem (A., Heaton, 2021)

Let $\mathcal{M}$ be a linear model. Then the logarithmic Voronoi cells are equal to their log-normal polytopes.

Any d-dimensional linear model inside Δ_{n-1} can be written as

$$\mathcal{M} = \{c - Bx : x \in \Theta\}$$

where B is a $n \times d$ matrix, whose columns sum to 0, and $c \in \mathbb{R}^n$ is a vector, whose coordinates sum to 1.

Discrete linear models

A *linear model* is given parametrically by nonzero linear polynomials.

Theorem (A., Heaton, 2021)

Let $\mathcal{M}$ be a linear model. Then the logarithmic Voronoi cells are equal to their log-normal polytopes.

Any d-dimensional linear model inside Δ_{n-1} can be written as

$$\mathcal{M} = \{c - Bx : x \in \Theta\}$$

where B is a $n \times d$ matrix, whose columns sum to 0, and $c \in \mathbb{R}^n$ is a vector, whose coordinates sum to 1.

A *co-circuit* of B is a vector $v \in \mathbb{R}^n$ of minimal support such that $vB = 0$. A co-circuit is *positive* if all its coordinates are positive.

We call a point $p = (p_1, \ldots, p_n) \in \mathcal{M}$ is *interior* if $p_i > 0$ for all $i \in [n]$.

Interior points

For an interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell at p is the set

$$\log \mathrm{Vor}_{\mathcal{M}}(p) = \left\{ r \cdot \mathrm{diag}(p) \in \mathbb{R}^n : rB = 0,\ r \geq 0,\ \sum_{i=1}^{n} r_i p_i = 1 \right\}.$$

Proposition (A.)

For any interior point $p \in \mathcal{M}$, the vertices of $\log \mathrm{Vor}_{\mathcal{M}}(p)$ are of the form $v \cdot \mathrm{diag}(p)$ where v are unique representatives of the positive co-circuits of B such that $\sum_{i=1}^{n} v_i p_i = 1$.

Gale diagrams

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a vector configuration in $\mathbb{R}^d$, whose affine hull has dimension d. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}.$$

Let $\{B_1, \ldots, B_{n-d-1}\}$ be a basis for $\ker(A)$ and $B := [B_1\ B_2\ \cdots\ B_{n-d-1}]$. The configuration $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ of row vectors of B is the *Gale diagram* of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

Gale diagrams

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a vector configuration in $\mathbb{R}^d$, whose affine hull has dimension d. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}.$$

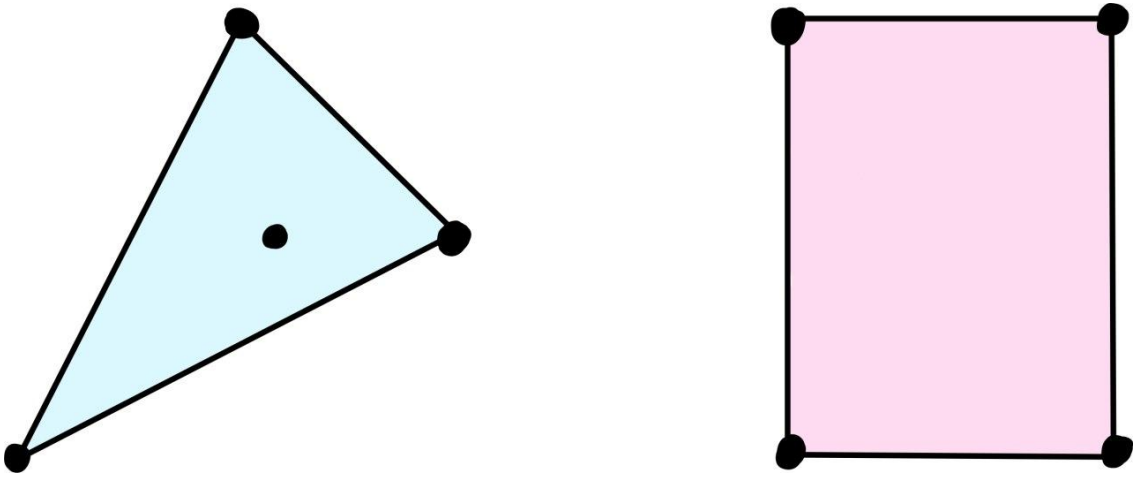
Let $\{B_1, \ldots, B_{n-d-1}\}$ be a basis for $\ker(A)$ and $B := [B_1\ B_2\ \cdots\ B_{n-d-1}]$. The configuration $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ of row vectors of B is the *Gale diagram* of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

Theorem (A.)

For any interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell at p is combinatorially isomorphic to the dual of the polytope obtained by taking the convex hull of a vector configuration with Gale diagram B.

Corollary

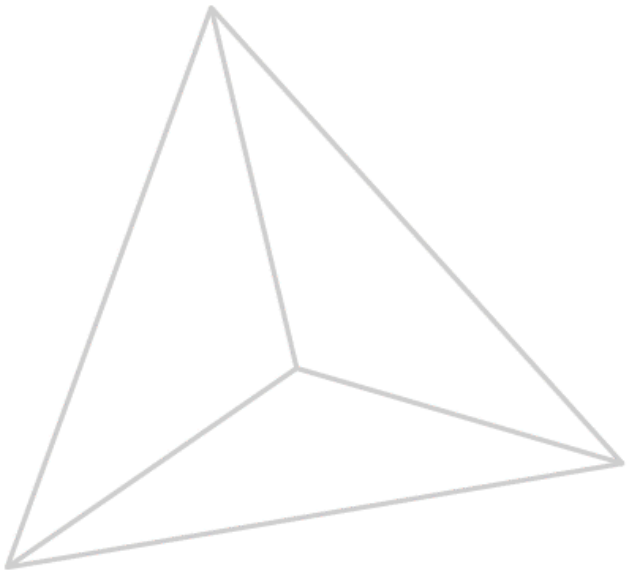
Logarithmic Voronoi cells of all interior points in a linear model have the same combinatorial type.



Proposition (A.)

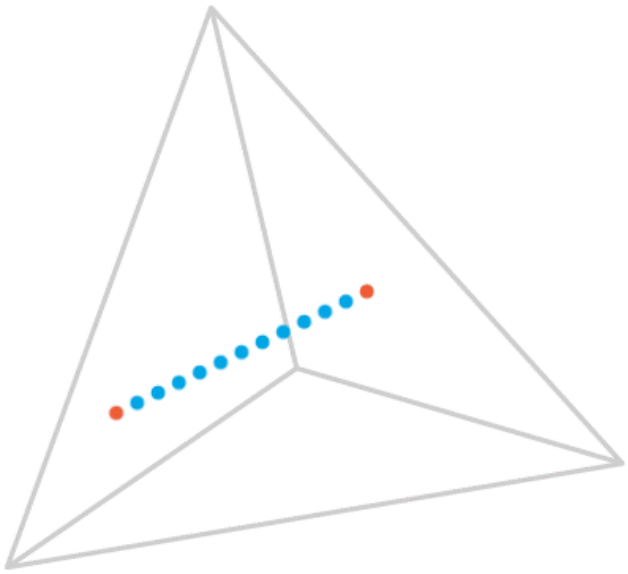
Every $(n-d-1)$*-dimensional polytope with at most* n *facets appears as a logarithmic Voronoi cell of a* d*-dimensional linear model inside* Δ_{n-1}.

Examples



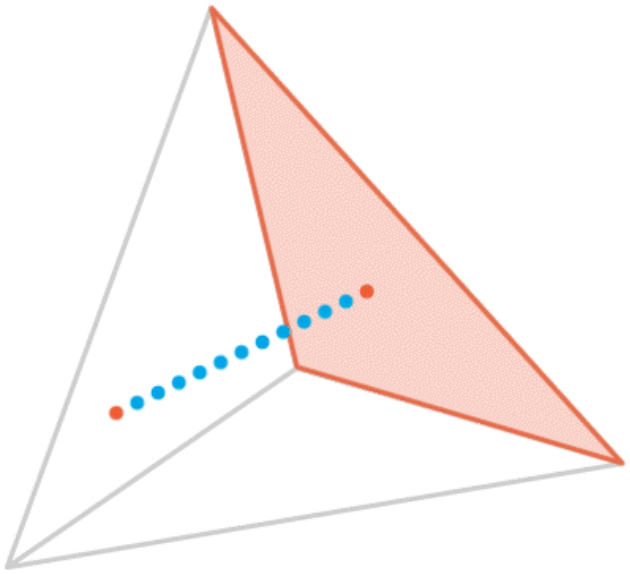
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



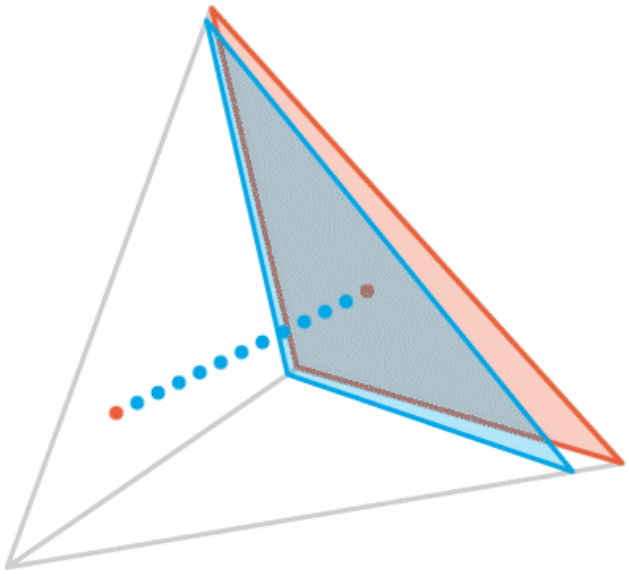
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



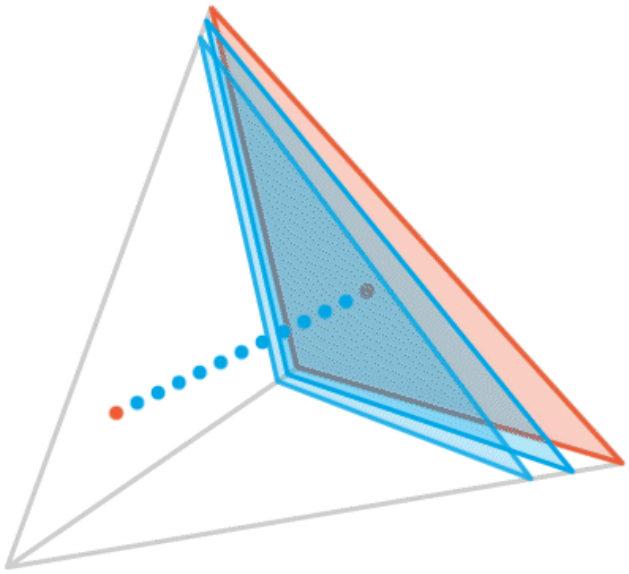
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



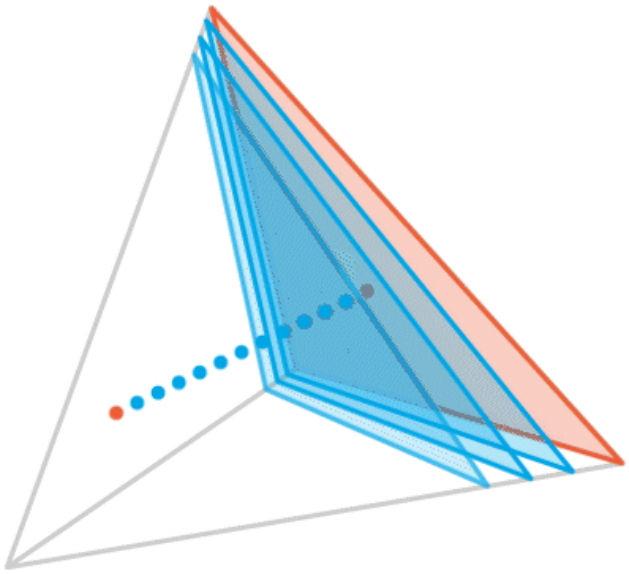
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



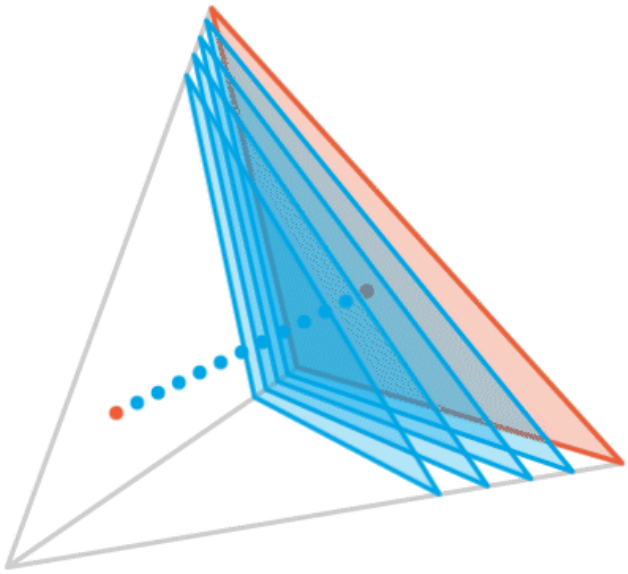
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



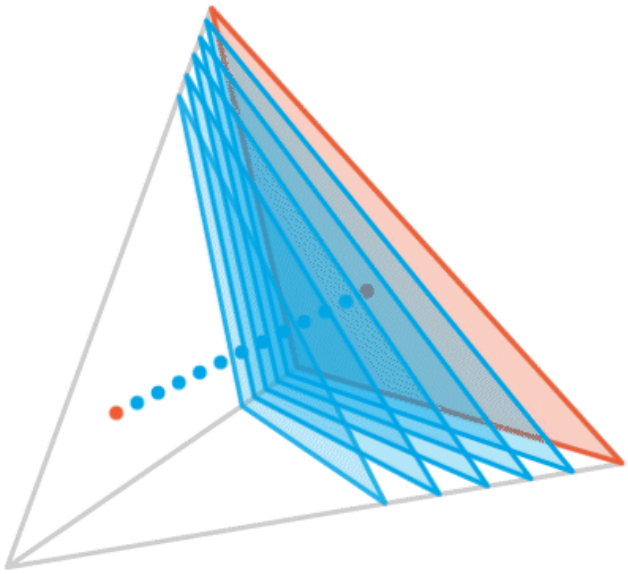
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



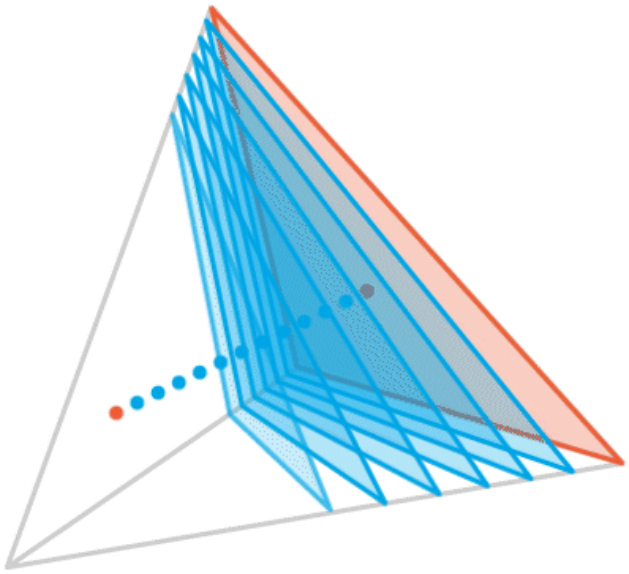
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



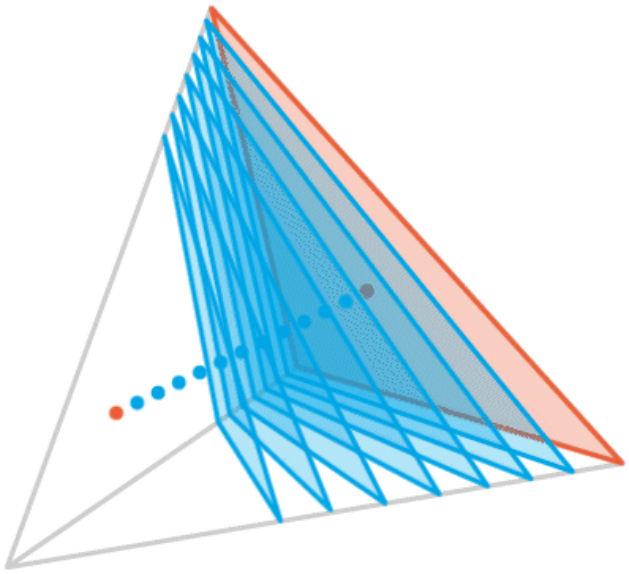
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



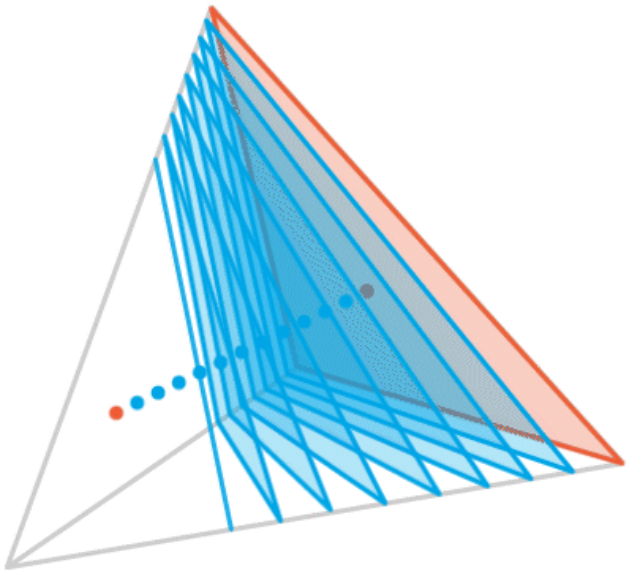
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



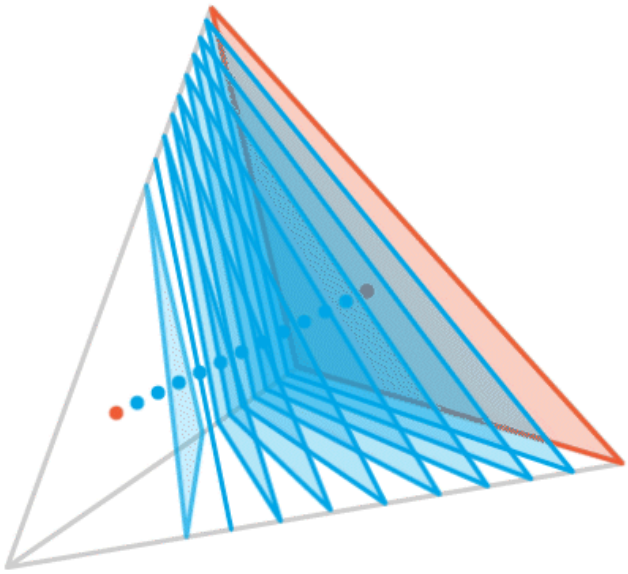
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



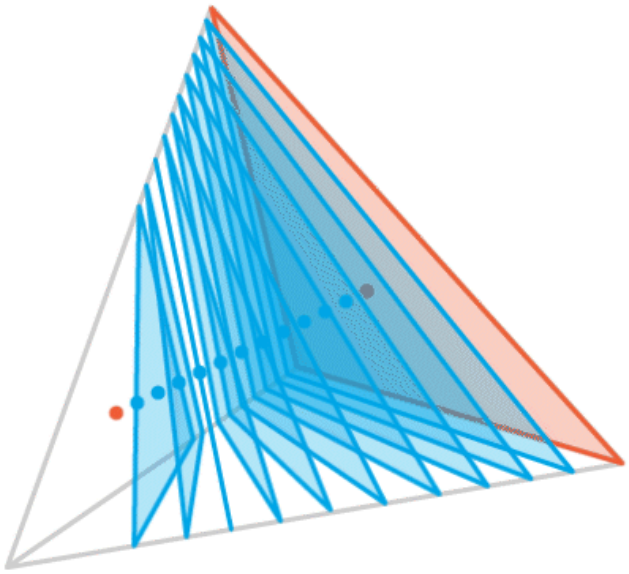
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



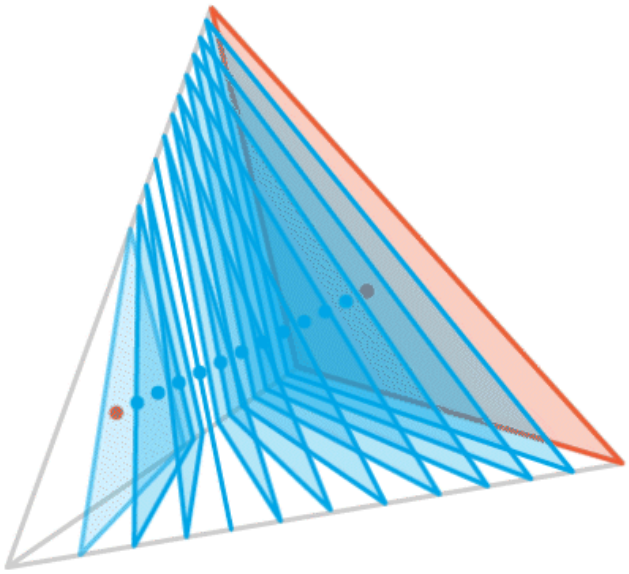
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



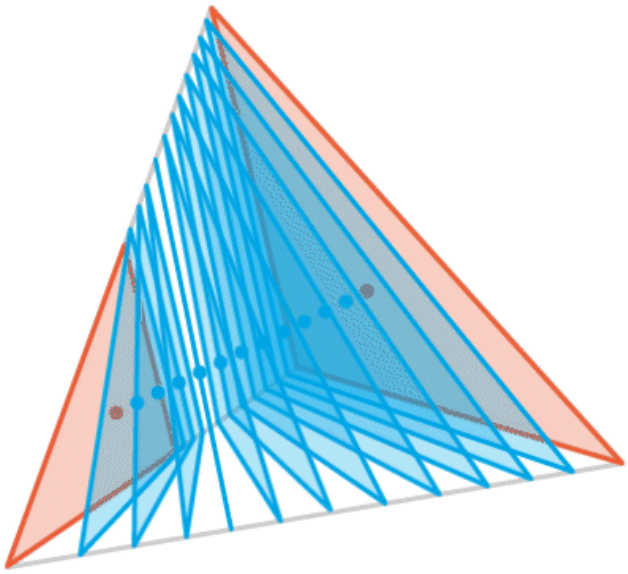
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



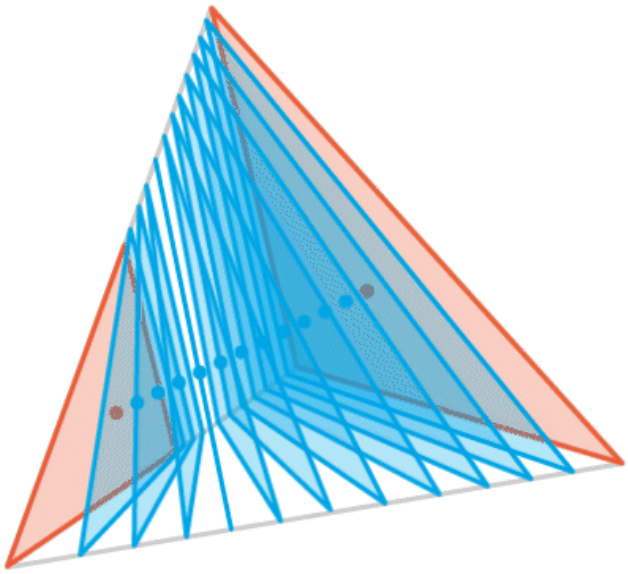
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



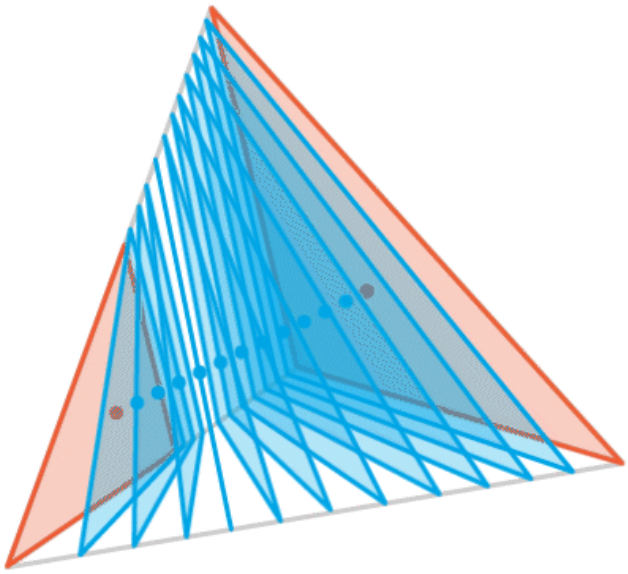
$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

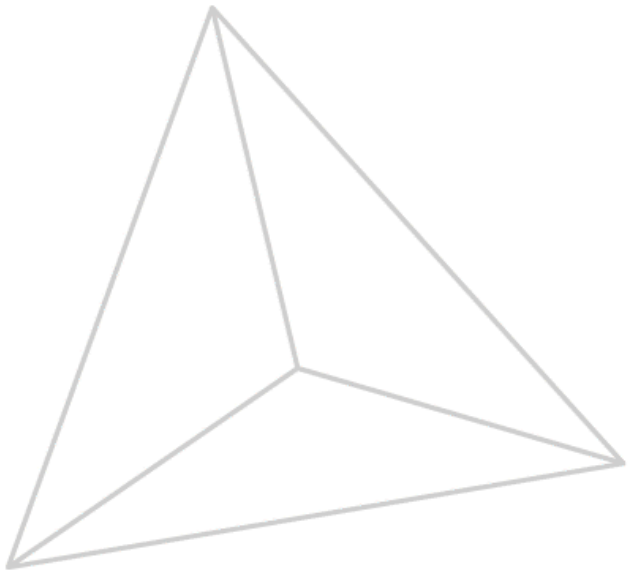


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

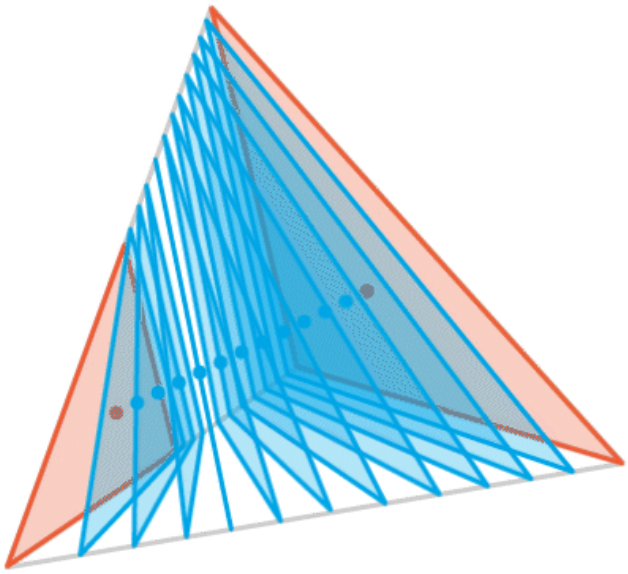


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

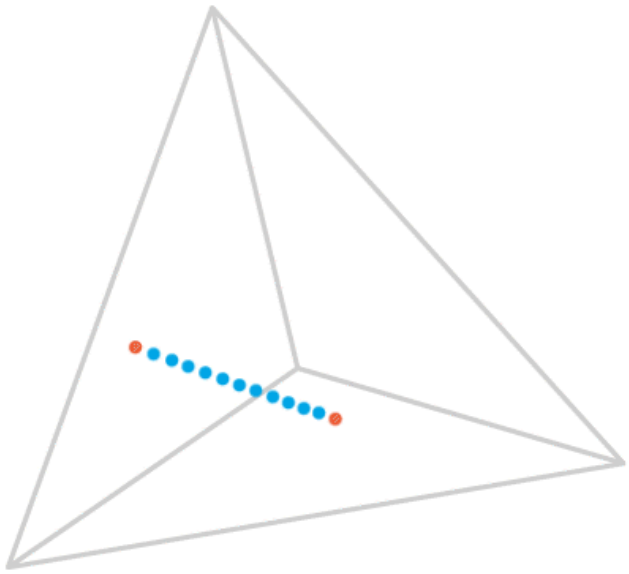


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

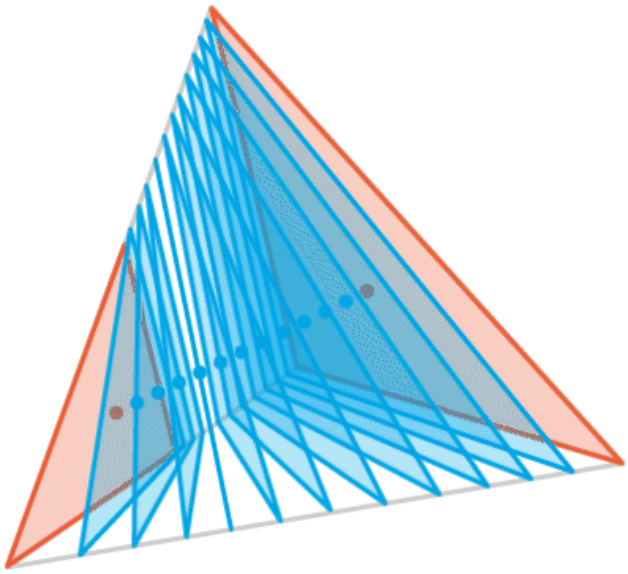


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

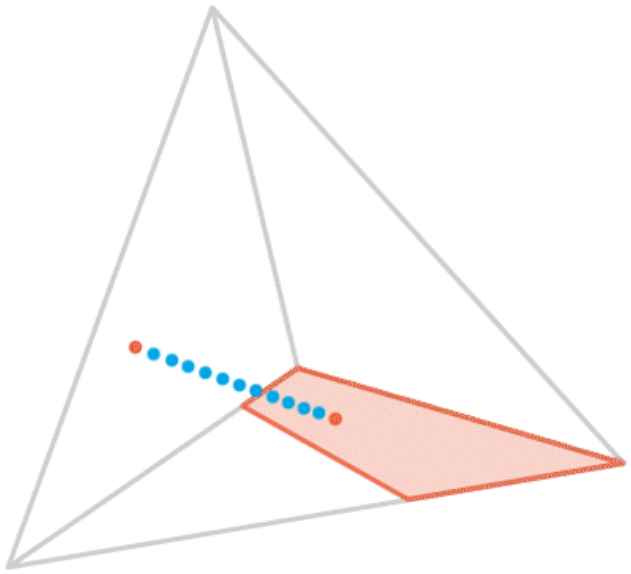


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

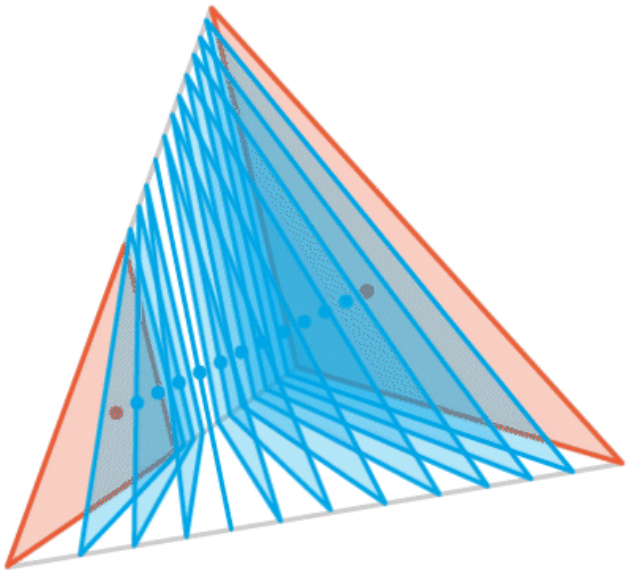


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

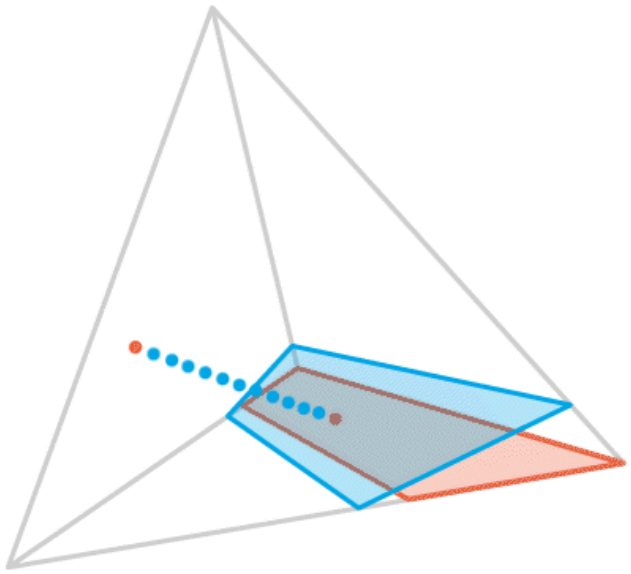


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

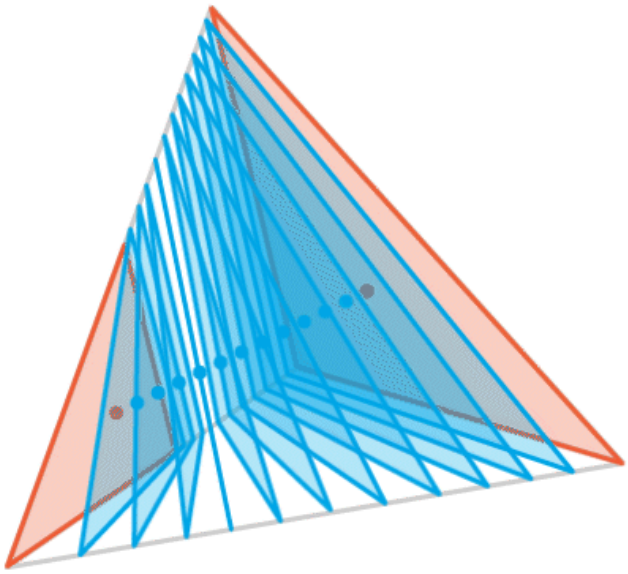


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

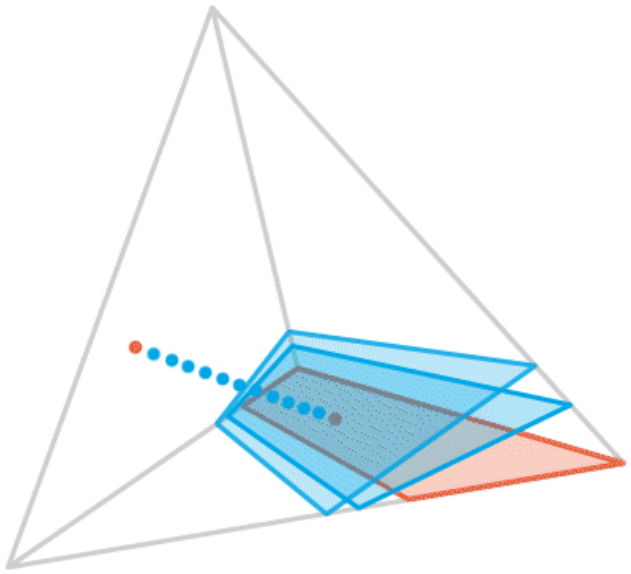


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

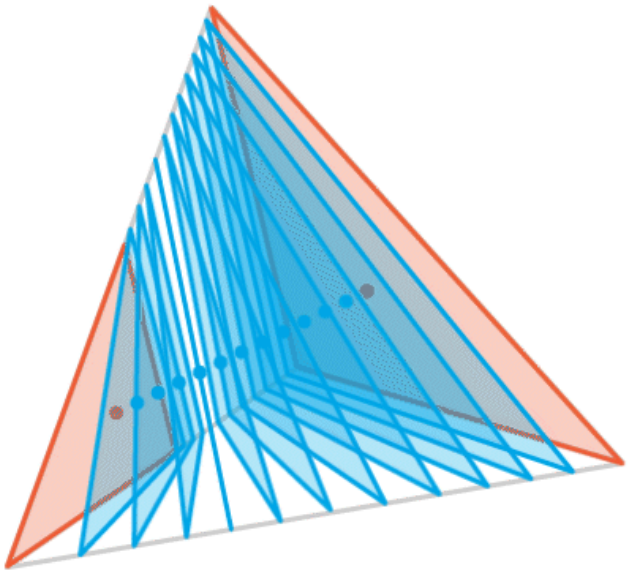


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

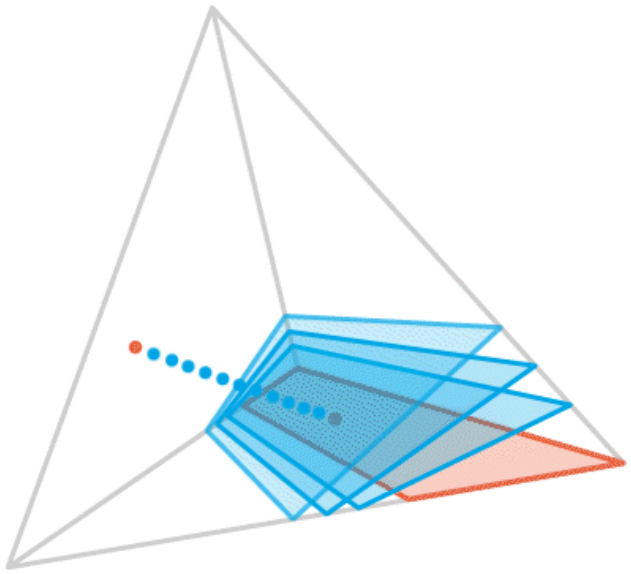


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

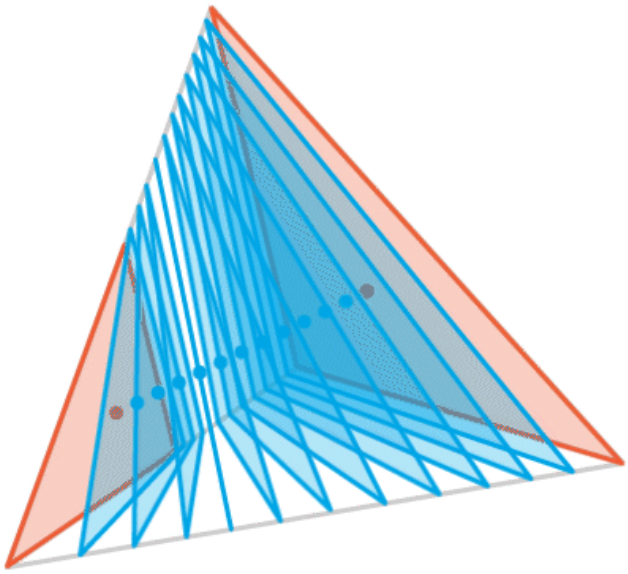


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

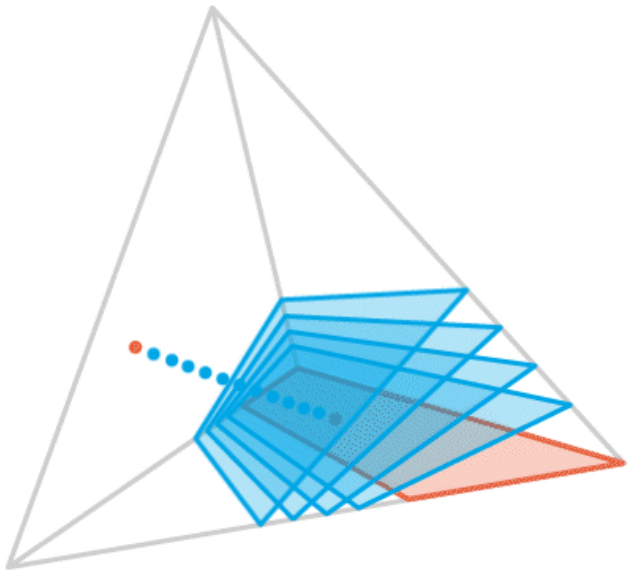


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

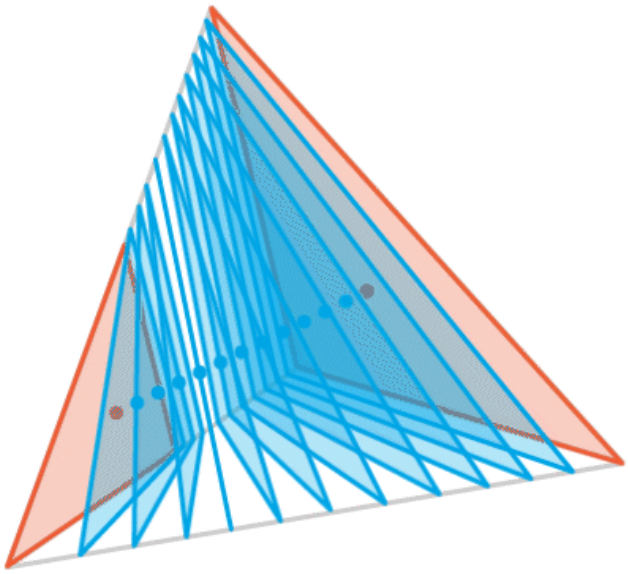


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

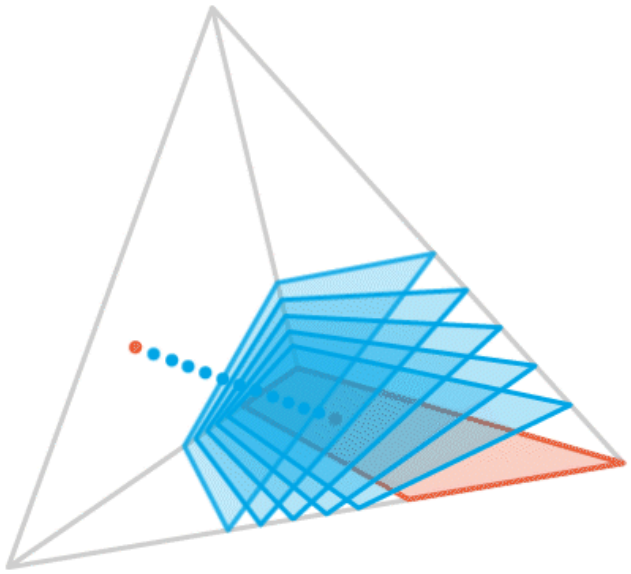


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

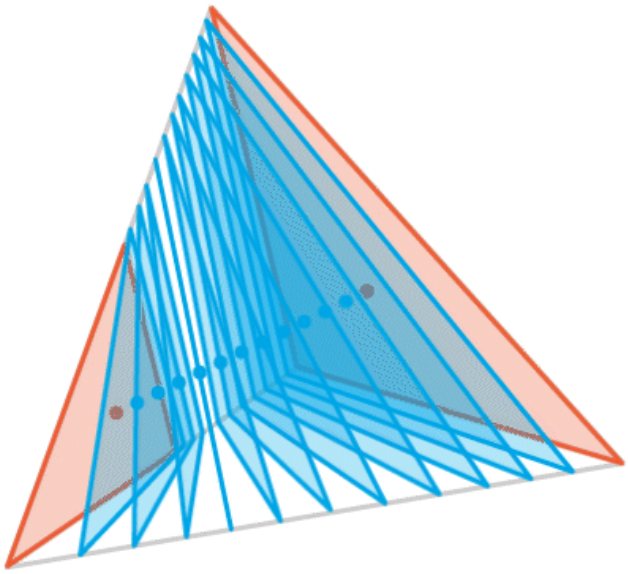


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

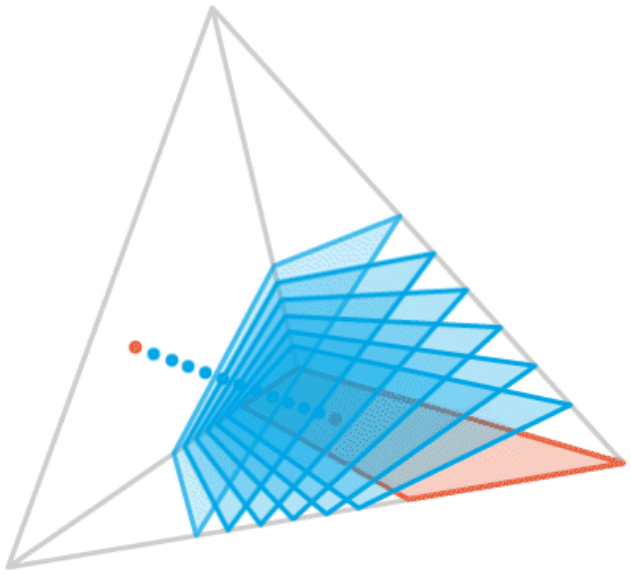


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

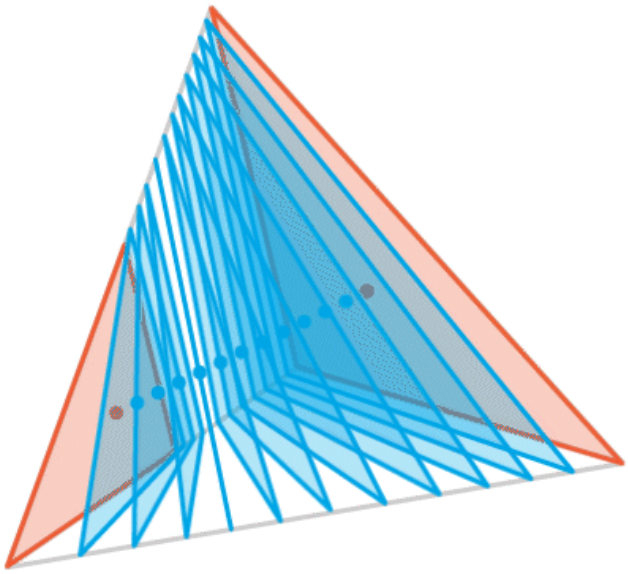


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

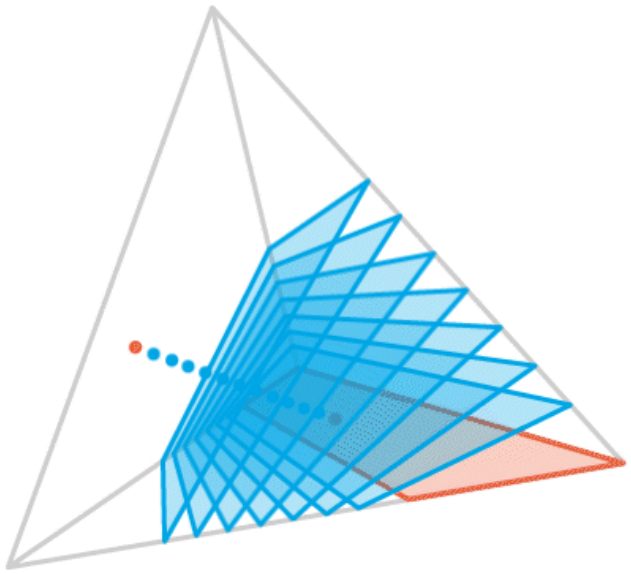


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

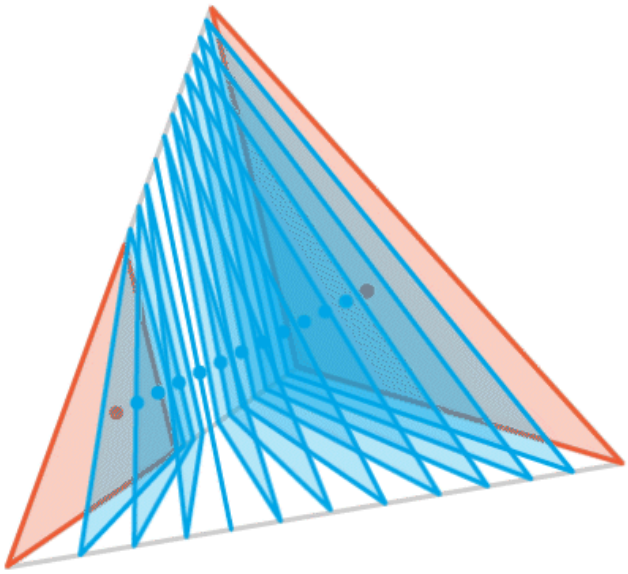


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

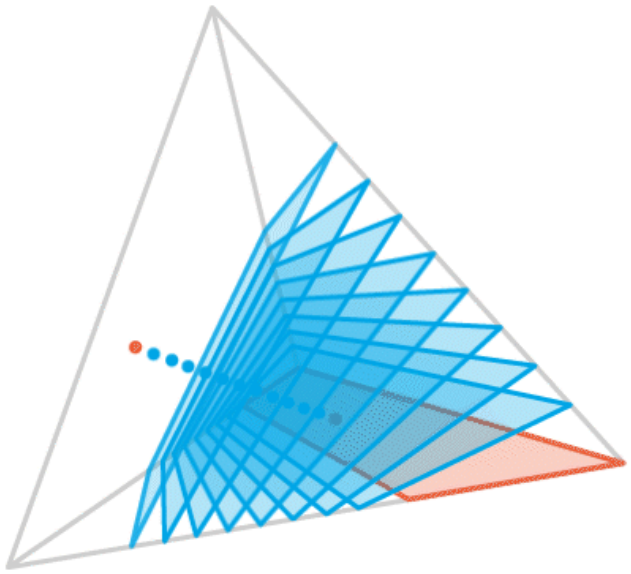


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

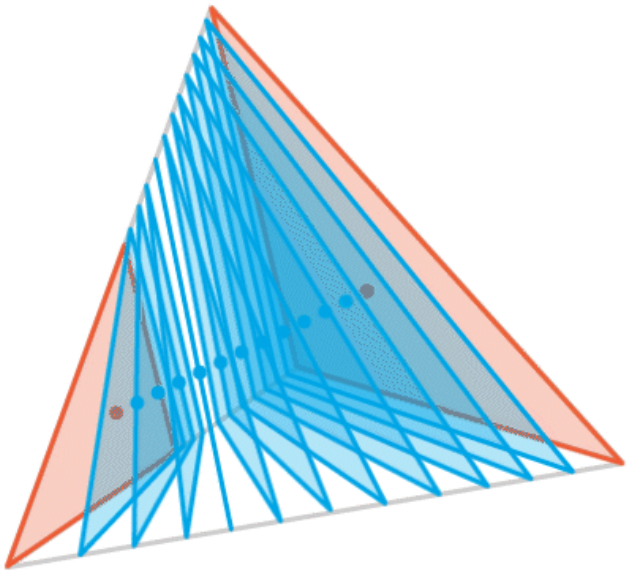


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

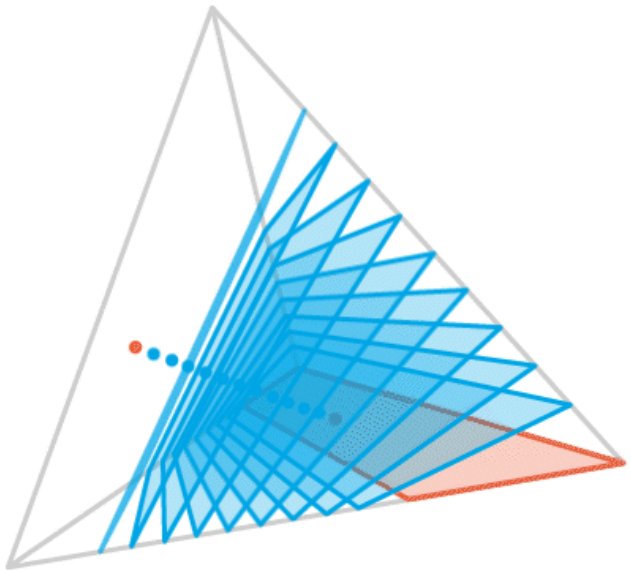


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

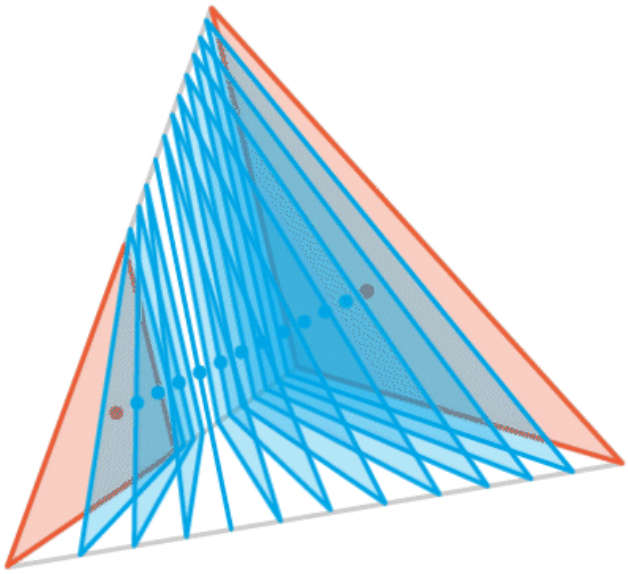


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

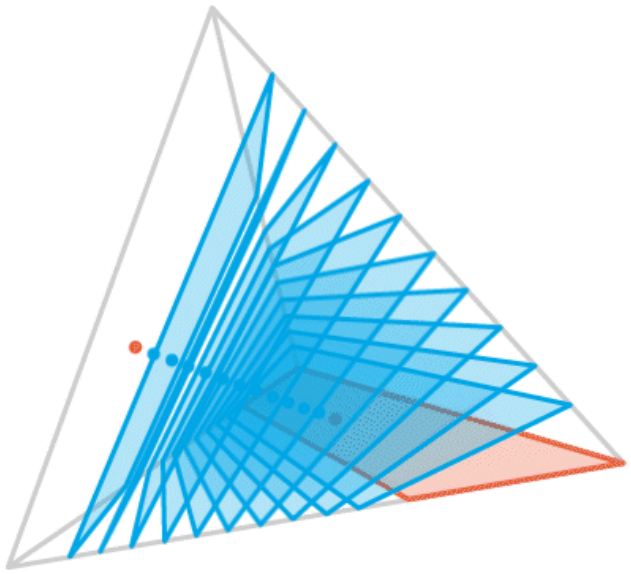


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

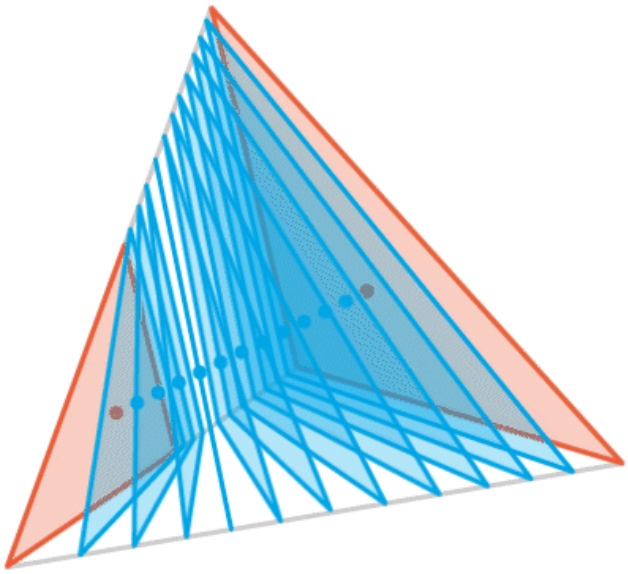


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

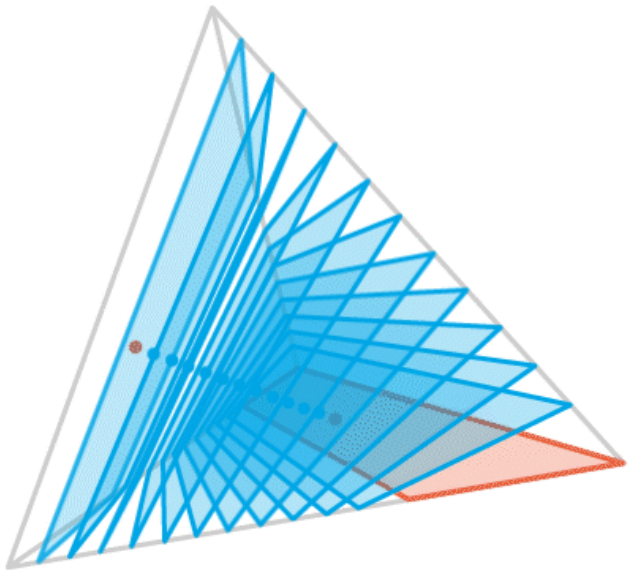


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples

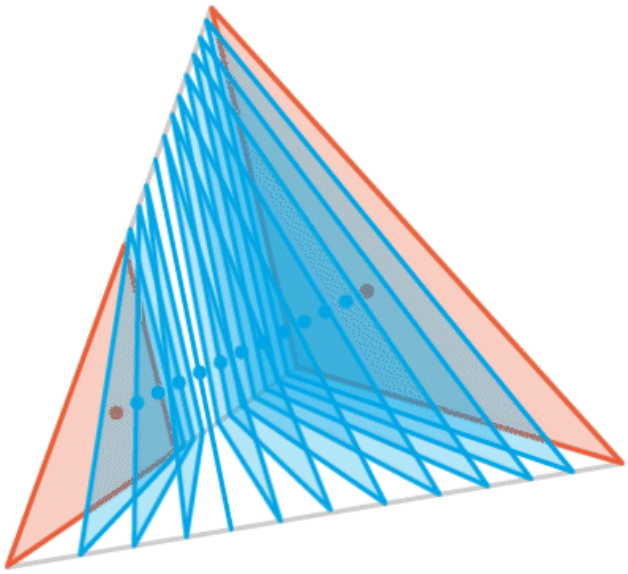


$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

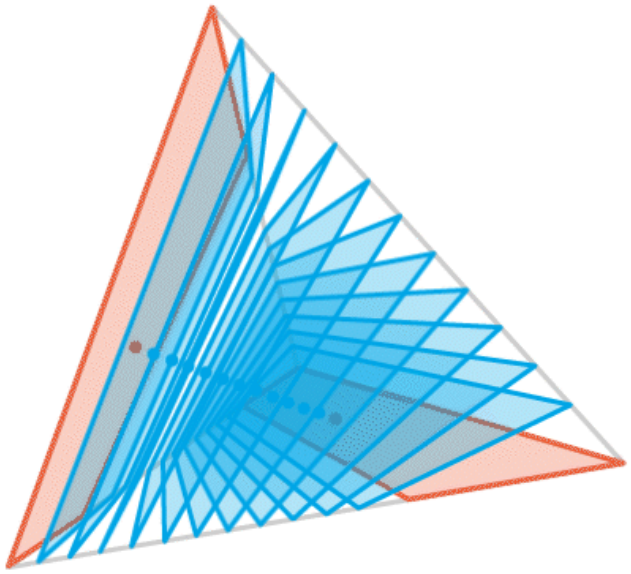


$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

Examples



$$B = [1, -5, 3, 1]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$



$$B = [-2, -1, 1, 2]^T$$
$$c = (1/4, 1/4, 1/4, 1/4)$$

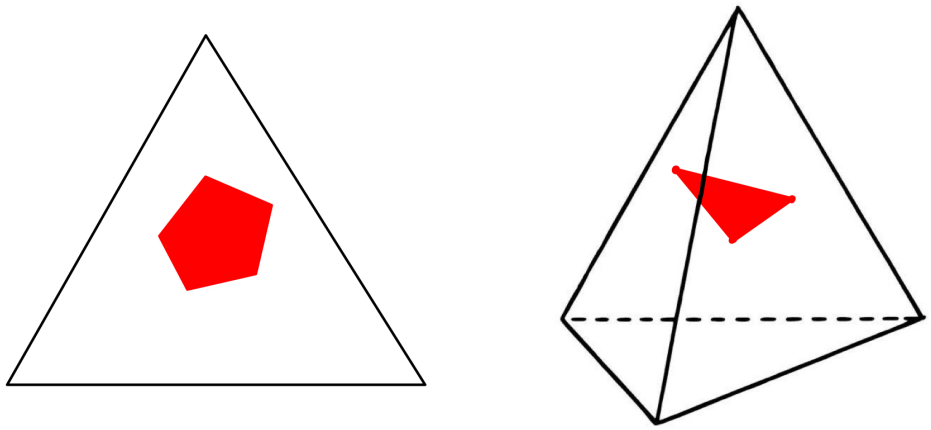
On the boundary

Theorem (A.)

Let $\mathcal{M}$ be the d-dimensional linear model, obtained by intersecting the affine linear space L with Δ_{n-1}. Let $w \in \mathcal{M}$ be a point on the boundary of the simplex. If L intersects Δ_{n-1} transversally, then the logarithmic Voronoi polytope at w has the same combinatorial type as those at the interior points of $\mathcal{M}$.

Partial linear models

A *partial linear model* of dimension d is a statistical model given by a d-dimensional polytope inside the probability simplex Δ_{n-1}, such that not all facets of the polytope lie on the boundary of the simplex.



The intersection of the affine span of the polytope $\mathcal{M}$ with the simplex Δ_{n-1} is a d-dimensional linear model $\mathcal{M}'$. We say that $\mathcal{M}'$ *extends* $\mathcal{M}$.

Parial linear models

Proposition (A.)

Let $\mathcal{M}$ be a partial linear model of dimension d with extension $\mathcal{M}'$. If p is a point in the relative interior of $\mathcal{M}$, then $\log \mathrm{Vor}_{\mathcal{M}}(p) = \log \mathrm{Vor}_{\mathcal{M}'}(p)$.

Let $p \in \partial\mathcal{M}$. Then $\log \mathrm{Vor}_{\mathcal{M}'}(p) \subseteq \log \mathrm{Vor}_{\mathcal{M}}(p)$, but in general this containment will be strict.

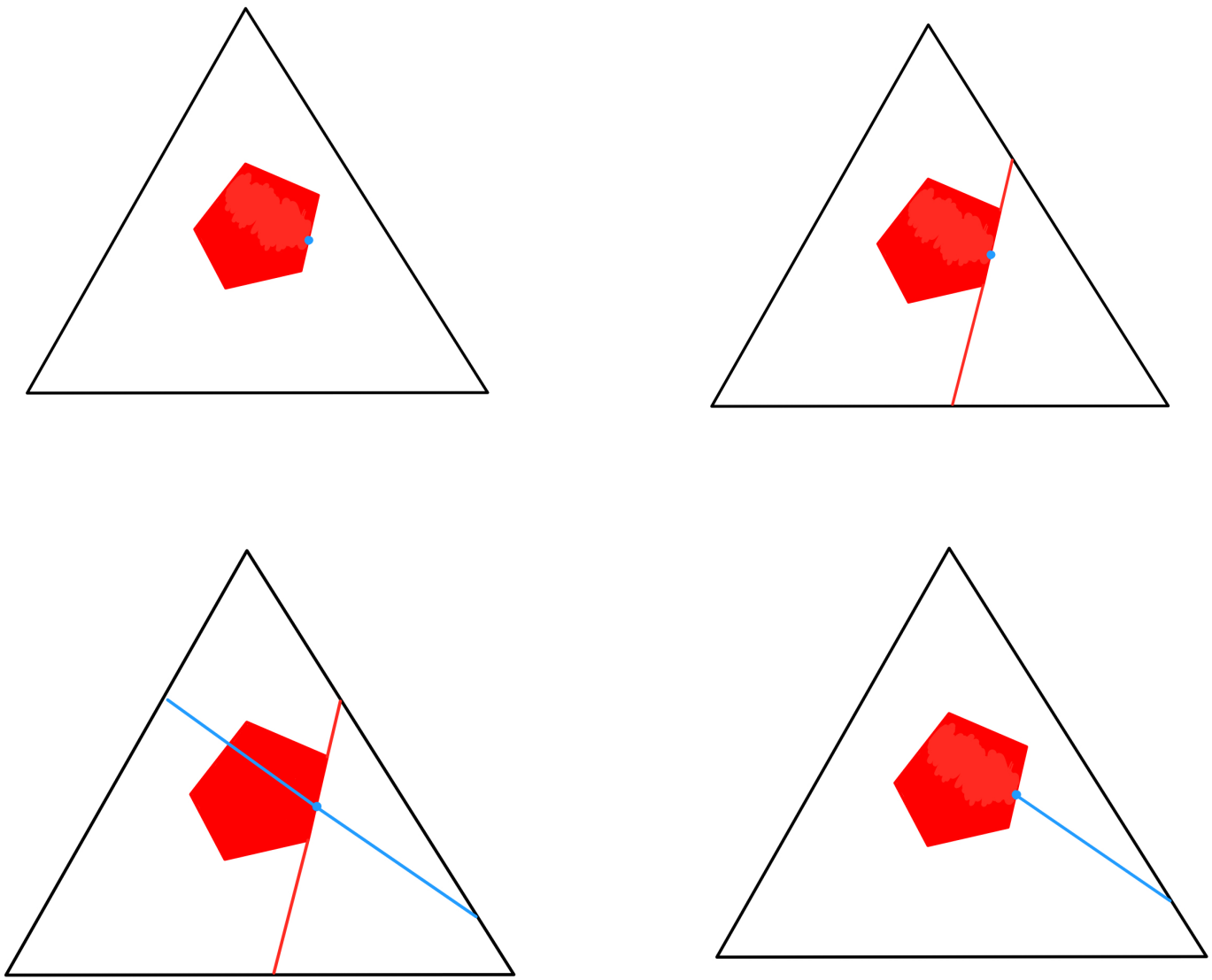
Let F be a facet of $\mathcal{M}$ and $p \in F^\circ$. How to compute $\log \mathrm{Vor}_{\mathcal{M}}(p)$?

- Treat F as its own partial linear model with extension F' inside Δ_{n-1}.
- Then $\log \mathrm{Vor}_F(p) = \log \mathrm{Vor}_{F'}(p)$ is an $(n-d)$-dimensional polytope.
- Note $\log \mathrm{Vor}_{\mathcal{M}'}(p)$ divides the polytope $\log \mathrm{Vor}_F(p)$ into two $(n-d)$-dimensional polytopes, Q_p and $\bar{Q}_p$.
- Only of those polytopes, $\bar{Q}_p$ will intersect the interior of $\mathcal{M}$.

Proposition (A.)

Let p be a point in the relative interior of some facet F of $\mathcal{M}$. Then $Q_p = \log \mathrm{Vor}_{\mathcal{M}}(p)$.

Partial linear models

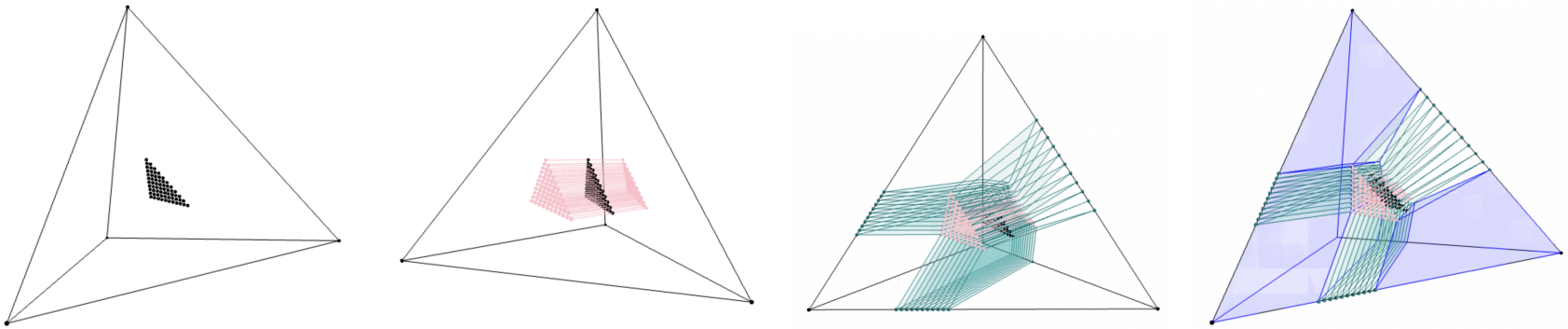


What about other faces?

- Let F is a face of $\mathcal{M}$ of dimension $d-k$ for some $k \geq 2$.
- F is the intersection of at least k faces $\{G_1, \cdots, G_m\}$ of dimension $d-k+1$.
- Each $\log \mathrm{Vor}_{G_i'}(p)$ subdivides $\log \mathrm{Vor}_{F'}(p)$ into two polytopes.
- One will intersect G_i at an interior point. Call the other polytope Q_i.

Conjecture

Let p be a point in the relative interior of a $(d-k)$-dimensional face F of $\mathcal{M}$ as above. Then $\bigcap_{i\in[m]} Q_i = \log \mathrm{Vor}_{\mathcal{M}}(p)$. In particular, if $\mathcal{M}$ is in general position, $\dim \log \mathrm{Vor}_{\mathcal{M}}(p) = (n-1) - \dim F$.



Mahalo!